ISRAEL JOURNAL OF MATHEMATICS, Vol. 34, Nos. 1-2, 1979

A REMARKABLE HOMOGENEOUS BANACH ALGEBRA

BY
G. PISIER

ABSTRACT
We give an example of a strongly homogeneous Banach algebra B with $A(\mathbf{T}) \subsetneqq B \subsetneqq C(\mathbf{T})$ on which all 1-Lipschitzian functions operate; this improves previous results of M. Zafran.

We give an example of a strongly homogeneous Banach algebra B, with $A(\mathbf{T}) \subsetneqq B \subsetneqq C(\mathbf{T})$ on which all Lipschitzian functions of order 1 operate. Recently, M. Zafran [9] obtained an example on which non-analytic functions operate; even more recently [10], he exhibited a strongly homogeneous Banach algebra on which all C^3 functions operate. Our example seems to be optimal with respect to the symbolic calculus: indeed, if there exists $F:[-1,+1] \to \mathbf{R}$ satisfying $F(0) = 0$ and $\lim_{t\to 0} |F(t)/t| = \infty$ and F operates on B, then necessarily $B = C(\mathbf{T})$ (see [5]).

Let $(g_n)_{n\in\mathbf{Z}}$ (resp. $(\tilde{g}_n)_{n\in\mathbf{Z}}$) be a sequence of independent, Gaussian, real (resp. complex) random variables, such that $\mathbf{E}|g_n|^2 = 1$ and $\mathbf{E}g_n = 0$ (resp. $\mathbf{E}|\tilde{g}_n|^2 = 1$ and $\mathbf{E}\tilde{g}_n = 0$). Let $(\varepsilon_n)_{n\in\mathbf{Z}}$ be a sequence of independent random variables with the same distribution and $\varepsilon_n = \pm 1$ with probability $\frac{1}{2}$. All the random variables are defined on some probability space $(\Omega, \mathcal{A}, \mathbf{P})$.

We will consider the linear space $C_{a.s}(\mathbf{T})$ which consists of all the functions f in $L^2(\mathbf{T})$ such that the random Fourier series

$$(1) \qquad \sum_{n\in\mathbf{Z}} \tilde{g}_n(\omega)\hat{f}(n)e^{int}$$

represents a continuous function for almost all ω in Ω. According to the known results on the integrability of the norms of Gaussian vectors (see [2]) we may equip $C_{a.s}(\mathbf{T})$ with the norm $\{\ \ \}$ defined by

Received November 24, 1978

$$\{f\} = \mathbf{E}\left\| \sum_{n\in\mathbf{Z}} \tilde{g}_n \hat{f}(n) e^{int} \right\|_\infty$$

(where we have denoted $\|.\|_\infty$ the norm in the space $C(\mathbf{T})$); with this norm, the space $C_{a.s}(\mathbf{T})$ becomes a Banach space.

REMARK 1. It is well known that if $f \in C_{a.s}(\mathbf{T})$ then $\mathbf{E}\|\Sigma_{|n|>N}\tilde{g}_n\hat{f}(n)e^{int}\|_\infty \to 0$ when $N \to \infty$, so that the set of all trigonometric polynomials is dense in $C_{a.s}(\mathbf{T})$. Therefore, f belongs to $C_{a.s}(\mathbf{T})$ iff the partial sums of the Fourier series of f form a Cauchy sequence in $C_{a.s}(\mathbf{T})$. (Actually, this happens as soon as these partial sums are bounded in $C_{a.s}(\mathbf{T})$, according to results of Belyaev and Billard; see [3] or [8] for details and references.)

REMARK 2. By the main result of [7], the series (1) is a.s. continuous if and only if the corresponding Bernoulli series $\Sigma_{n\in\mathbf{Z}}\varepsilon_n(\omega)\hat{f}(n)e^{int}$ is a.s. continuous (such series have been extensively studied in [4]). Therefore all the statements of this note are true if we replace everywhere $(\tilde{g}_n)_{n\in\mathbf{Z}}$ by $(\varepsilon_n)_{n\in\mathbf{Z}}$, only the constants appearing in the corresponding inequalities have to be changed. The same is true if we replace $(\tilde{g}_n)_n$ by $(g_n)_n$, but this is elementary.

REMARK 3. (i) Using the invariance properties of the distribution of $(\tilde{g}_n)_{n\in\mathbf{Z}}$, it is easy to check that the mappings $f(x) \to f(x+a)$ and $f(x) \to f(kx)$ are isometric on $C_{a.s}(\mathbf{T})$, for every a in $\mathbf{T}$ and every integer $k \neq 0$. Also, $\forall f \in C_{a.s}(\mathbf{T})$, $\bar{f} \in C_{a.s}(\mathbf{T})$ and $\{\bar{f}\} = \{f\}$ so that $C_{a.s}(\mathbf{T})$ is self-adjoint.

(ii) Actually, if $f, h \in C_{a.s}(\mathbf{T})$ and if $\forall n \in \mathbf{Z}$, $|\hat{f}(n)| \leqq |\hat{h}(n)|$, then clearly $\{f\} \leqq \{h\}$.

We come to our main result. Consider the space $B = C(\mathbf{T}) \cap C_{a.s}(\mathbf{T})$, equipped with the norm $|\|\ \ \|\|$ defined by

$$\forall f \in B, \qquad |\|f\|| = 9\sqrt{2}\|f\|_\infty + \{f\}.$$

B is again a Banach space in which the set of all trigonometric polynomials is dense, and also $A(\mathbf{T}) \subsetneqq B \subsetneqq C(\mathbf{T})$ and $\forall f \in A(\mathbf{T})$

$$(9\sqrt{2})\|f\|_\infty \leqq |\|f\|| \leqq (1 + 9\sqrt{2})\|f\|_{A(\mathbf{T})}.$$

But much more is true:

THEOREM. (i) $B = C(\mathbf{T}) \cap C_{a.s}(T)$ *is a Banach algebra under pointwise multiplication*; *in fact, it is a strongly homogeneous Banach algebra.*

(ii) *All the* 1-*Lipschitzian functions operate on* B.

The two points of the theorem are simple consequences of a principle of comparison for Gaussian processes which originates in a lemma of Slepian; the following variant was proposed by Sudakov (see [2] for the proof and related details):

LEMMA 1. *Let* $(X_t)_{t\in S}$, $(Y_t)_{t\in S}$ *be real Gaussian processes on a set* S, *defined on* $(\Omega,\mathbf{P})$. *If* $\forall t,s\in S$, $\mathbf{E}|Y_t - Y_s|^2 \leqq \mathbf{E}|X_t - X_s|^2$ *then*

$$\mathbf{E}\sup_{t\in S} Y_t \leqq \mathbf{E}\sup_{t\in S} X_t$$

(*where* $\sup_{t\in S} X_t$ *means the supremum in the lattice* $L^1(d\mathbf{P})$).

REMARK 4. Let S be a compact metric space and let (X_t) and (Y_t) be as above; then if (X_t) is a.s. continuous on S, so is (Y_t). This fact was proved in [6] using the original lemma of Slepian (which is the same as Lemma 1 above, with the additional assumption that $\mathbf{E}|X_t|^2 = \mathbf{E}|Y_t|^2$, $\forall t\in S$). Actually, with the same argument as in lemma 2.1 of [6], one can deduce the preceding lemma from Slepian's lemma, except that the conclusion is only

$$\mathbf{E}\sup Y_t \leqq 2\mathbf{E}\sup X_t,$$

which is sufficient for most applications.

REMARK 5. For f in $L^2(\mathbf{T})$ and t in $\mathbf{T}$ we write $f_t(x) = f(t+x)$ and

$$\tilde{X}_t(\omega) = \sum_{n\in\mathbf{Z}} \tilde{g}_n(\omega)\hat{f}(n)e^{int}.$$

Let $(\tilde{X}'_t)_{t\in\mathbf{T}}$ be an independent copy of the process $(\tilde{X}_t)_{t\in\mathbf{T}}$. We denote $\|\ \|_2$ the norm in $L^2(\mathbf{T})$. To each f in $L^2(\mathbf{T})$, we associate the *real* Gaussian process (X_t) defined by

$$X_t(\omega) = \operatorname{Re}\tilde{X}_t(\omega) + \operatorname{Im}\tilde{X}'_t(\omega).$$

Clearly we have, $\forall s,t\in\mathbf{T}$,

$$\mathbf{E}|X_t - X_s|^2 = \mathbf{E}|\tilde{X}_t - \tilde{X}_s|^2 = \|f_t - f_s\|_2^2.$$

To translate the information of Lemma 1 in terms of random Fourier series, we will need some easily checked inequalities.

First note that

$$\mathbf{E}\sup_t X_t \leqq 2\mathbf{E}\sup_t |\tilde{X}_t| = 2\{f\}. \tag{2}$$

On the other hand

$$\mathbf{E}\sup_t |\tilde{X}_t - \tilde{X}_0| \leqq \mathbf{E}\sup_t |\mathrm{Re}(\tilde{X}_t - \tilde{X}_0)| + \mathbf{E}\sup_t |\mathrm{Im}(\tilde{X}_t - \tilde{X}_0)|$$

and since $\mathrm{Re}(\tilde{X}_t - \tilde{X}_0)$ and $\mathrm{Im}(\tilde{X}'_t - \tilde{X}'_0)$ are obtained from $X_t - X_0$ by taking a suitable conditional expectation we must have

$$\begin{aligned}
\mathbf{E}\sup_t |\tilde{X}_t - \tilde{X}_0| &\leqq 2\mathbf{E}\sup_t |X_t - X_0| \\
&\leqq 2\mathbf{E}\sup_{t,s} |X_t - X_s| \\
&= 2\mathbf{E}\sup_{t,s} X_t - X_s \\
&= 2\mathbf{E}\sup_t X_t + 2\mathbf{E}\sup_s - X_s \\
&= 4\mathbf{E}\sup X_t \qquad \text{(by symmetry).}
\end{aligned}$$

Hence, $\{f\} = \mathbf{E}\sup|\tilde{X}_t| \leqq 4\mathbf{E}\sup X_t + \mathbf{E}|\tilde{X}_0|$ so that

$$\{f\} \leqq 4\mathbf{E}\sup_t X_t + \|f\|_2. \tag{3}$$

Now Lemma 1 yields immediately

LEMMA 2. *Consider f, h in $L^2(\mathbf{T})$ such that $\forall t, s \in \mathbf{T}$, $\|h_t - h_s\|_2 \leqq \|f_t - f_s\|_2$. Then if f belongs to $C_{a.s}(\mathbf{T})$, so does h. Moreover*

$$\{h\} \leqq 8\{f\} + \|h\|_2. \tag{4}$$

PROOF. By Remark 1, it is sufficient to prove (4) for f, h in $C_{a.s}(\mathbf{T})$. Let X_t be as above; define similarly a real Gaussian process (Y_t) associated with h so that

$$\forall t, s \in \mathbf{T}, \qquad \mathbf{E}|Y_t - Y_s|^2 = \|h_t - h_s\|_2^2.$$

By Lemma 1, we have

$$\mathbf{E}\sup Y_t \leqq \mathbf{E}\sup X_t, \tag{5}$$

and by (3)

$$\{h\} \leqq 4\mathbf{E}\sup_t Y_t + \|h\|_2,$$

so that (5) and (2) imply

$$\{h\} \leqq 8\{f\} + \|h\|_2. \qquad \text{q.e.d.}$$

PROOF OF THE THEOREM. (i) By Remark 3(i), it remains only to prove that B is stable under pointwise multiplication. So let $f, h \in B$. We have, $\forall t, s \in \mathbf{T}$,

$$\begin{aligned}\|(fh)_t - (fh)_s\|_2 &= \|f_t(h_t - h_s) + h_s(f_t - f_s)\|_2 \\ &\leqq \|f\|_\infty \|h_t - h_s\|_2 + \|h\|_\infty \|f_t - f_s\|_2 \\ &\leqq \|k_t - k_s\|_2\end{aligned}$$

where $k \in L^2(\mathbf{T})$ is defined by its Fourier coefficients, $\forall n \in \mathbf{Z}$, $\hat{k}(n) = \sqrt{2}(\|f\|_\infty^2 |\hat{h}(n)|^2 + \|h\|_\infty^2 |\hat{f}(n)|^2)^{1/2}$.

Applying Lemma 2, we obtain

$$\begin{aligned}\{fh\} &\leqq 8\{k\} + \|fh\|_2 \\ &\leqq 8\{k\} + \|f\|_\infty \|h\|_2.\end{aligned}$$

By Remark 3(ii) and the triangle inequality

$$\{k\} \leqq \sqrt{2}\|f\|_\infty \{h\} + \sqrt{2}\|h\|_\infty \{f\};$$

by the classical properties of complex Gaussian variables

$$\|h\|_2 = \frac{2}{\sqrt{\pi}}\mathbf{E}\,|\Sigma\,\hat{h}(n)\tilde{g}_n| \leqq \frac{2}{\sqrt{\pi}}\{h\} \leqq \sqrt{2}\{h\},$$

so that we conclude

$$\{fh\} \leqq 9\sqrt{2}(\|f\|_\infty \{h\} + \|h\|_\infty \{f\})$$

and therefore

$$|||\, fh\, ||| \leqq |||\, f\, ||| \cdot |||\, h\, |||.$$

(ii) Assume that $f \in B$ and $-1 \leqq f \leqq 1$, let $F:[-1,+1] \to \mathbf{C}$ be such that

$$\forall x, y \in [-1,+1], \qquad |F(x) - F(y)| \leqq |x-y| \quad \text{and} \quad |F(x)| \leqq 1.$$

Then, if $h = F \circ f$, we have obviously

$$\|h_t - h_s\|_2 \leqq \|f_t - f_s\|_2 \quad \text{and} \quad \|h\|_2 \leqq 1;$$

by Lemma 2, $h \in \mathrm{C}_{a.s}(\mathbf{T})$ therefore $h \in B$ and

$$\begin{aligned} |||\, h \,||| &= 9\sqrt{2}\|h\|_\infty + \{h\} \\ &\leqq 9\sqrt{2} + 8\{f\} + 1. \end{aligned}$$

REMARK 6. The proof of the first part of the theorem was inspired by S. Chevet's results [1] on tensor products of abstract Wiener spaces.

REMARK 7. The constants appearing in the definition of $|||\cdot|||$ indicate that a more natural norm might have been chosen. Indeed, with the notations of Remark 5, we may define

$$|f| = \sqrt{2}\|f\|_\infty + \mathbf{E}\sup_{t \in \mathbf{T}} X_t;$$

it is then easy to check that $|\ \ |$ is a strongly homogeneous Banach algebra norm on B (equivalent to $|||\ \ |||$). Note that if f is a constant then $\mathbf{E}\sup_{t\in\mathbf{T}} X_t = 0$, so that $|\ \ |$ behaves roughly like the norm in some space of Lipschitzian functions.

REMARK 8. The preceding theorem was essentially noticed in remark 3.2 in [8]. Note, however, that we have corrected a slight error: the first two lines on page 8 of [8] should be replaced by the above Lemma 2 and the last two lines of page 5 should be similarly corrected (these changes require one only to modify some numerical constants in [8]).

Finally, we mention that the preceding results remain clearly true on any compact abelian group. The reader is refered to [8] for a detailed discussion of the space $C_{a.s}(\mathbf{T})$.

Added in proof. J. P. Kahane kindly pointed out to me that the first part of the theorem follows from the second one, simply because $2fg = (f+g)^2 - (f-g)^2$ and $x \to x^2$ is 1-Lipschitzian at the origin.

REFERENCES

1. S. Chevet, *Un résultat sur les mesures gaussiennes*, C. R. Acad. Sci. Paris **A284** (1977), 441–444.

2. X. Fernique, *Régularité des trajectoires des fonctions aléatoires gaussiennes*, Springer Lecture Notes **480**, 1975.

3. N. Jain and M. Marcus, *Continuity of sub-Gaussian processes. Probability on Banach spaces*, Advances in Probability and Related Topics **4** (1978), 81–196.

4. J. P. Kahane, *Some random series of functions*, h.m.m. Lexington Press, 1968.

5. Y. Katznelson, *A characterization of all continuous functions on a compact Hausdorff space*, Bull. Amer. Math. Soc. **66** (1960), 313–315.

6. M. Marcus and L. Shepp, *Sample behavior of Gaussian processes*, Proc. Sixth Berkeley Symposium, 1970.

7. M. Marcus and G. Pisier, *Random Fourier series with applications to harmonic analysis*, to appear.

8. G. Pisier, *Sur l'espace de Banach des séries de Fourier aléatoires presque sûrement continues*, Exposé 17–18, Séminaire sur la Géométrie des Espaces de Banach, Ecole Polytechnique, Palaiseau, 1977–78.

9. M. Zafran, *The dichotomy problem for homogeneous Banach algebras*, Ann. Math. **108** (1978), 97–105.

10. M. Zafran, *On the symbolic calculus in homogeneous Banach algebras*, Israel J. Math. **32** (1979), 183–192.

CENTRE DE MATHEMATIQUES DE L'ECOLE POLYTECHNIQUE
PLATEAU DE PALAISEAU
91128 PALAISEAU CEDEX, FRANCE